Structured Electronic Design

EE3C11
Physics and modeling
of
Linear time-invariant dynamic systems

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Dynamic systems

A system:

- In which energy is stored
- Which is not in its quiescent state
- To which no excitations are applied

Tends to its quiescent state in which the energy is

- Distributed over system parts
- Dissipated in system parts
- Radiated by system parts

If excitations are applied to such systems

- Responses (output signals) depend on:
  - Energy stored in the past (past values of the excitations)
  - Current values of the excitations

Such systems are called dynamic systems

Linear(ized) time-invariant dynamic systems

Modeling with linear differential equations with constant coefficients

Sum of a number of derivatives of the response

$$\sum_{i=0}^{i=n} a_i \frac{d^i y(t)}{dt^i} = \sum_{k=0}^{k=m} b_k \frac{d^k x(t)}{dt^k}$$

Sum of a number of derivatives of the excitation

Exponential functions retain their shape under differentiation and integration

Fourier: write signals as sum of imaginary exponentials (finite energy signals only)

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) \exp(j\omega t) d\omega$$

$$X(j\omega) = \mathcal{F}\{x(t)\}$$

Complex amplitude of the imaginary exponential

Fourier transform of $x(t)$

Laplace: write signals as sum of complex exponentials

$$x(t) = \frac{1}{2\pi j} \oint_{\sigma - j\omega}^{\sigma + j\omega} X(s) \exp(st) ds$$

$$X(s) = \mathcal{L}\{x(t)\}$$

Complex amplitude of the complex exponential

Laplace transform of $x(t)$

Linear(ized) time-invariant dynamic systems

Differential equation changes into algebraic equation

Transfer function relates response of each exponential
to the excitation of the corresponding exponential

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{k=m} b_k (j\omega)^k}{\sum_{i=0}^{i=n} a_i (j\omega)^i}$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{k=m} b_k s^k}{\sum_{i=0}^{i=n} a_i s^i}$$

Poles and Zeros

Transfer function: $$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{k=m} b_k s^k}{\sum_{i=0}^{i=n} a_i s^i} = \frac{b_m \prod_{k=0}^{k=m} (s - z_k)}{a_n \prod_{i=0}^{i=n} (s - p_i)}$$

Poles are the solutions of the characteristic equation: $\sum_{i=0}^{i=n} a_i s^i = 0$

Stems from homogeneous differential equation: $\sum_{i=0}^{i=n} a_i \frac{d^i y(t)}{dt^i} = 0$ no excitations

Nonzero output signal in the absence of an excitation: result of energy storage.

Zeros are complex frequencies at which there is no signal transfer: $\sum_{k=0}^{k=m} b_k s^k = 0$

Poles and zeros are real or pairs of complex conjugates.

They constitute the real coefficients of the differential equation.

Companion matrix

Describe the characteristic equation as a set of first-order equations:

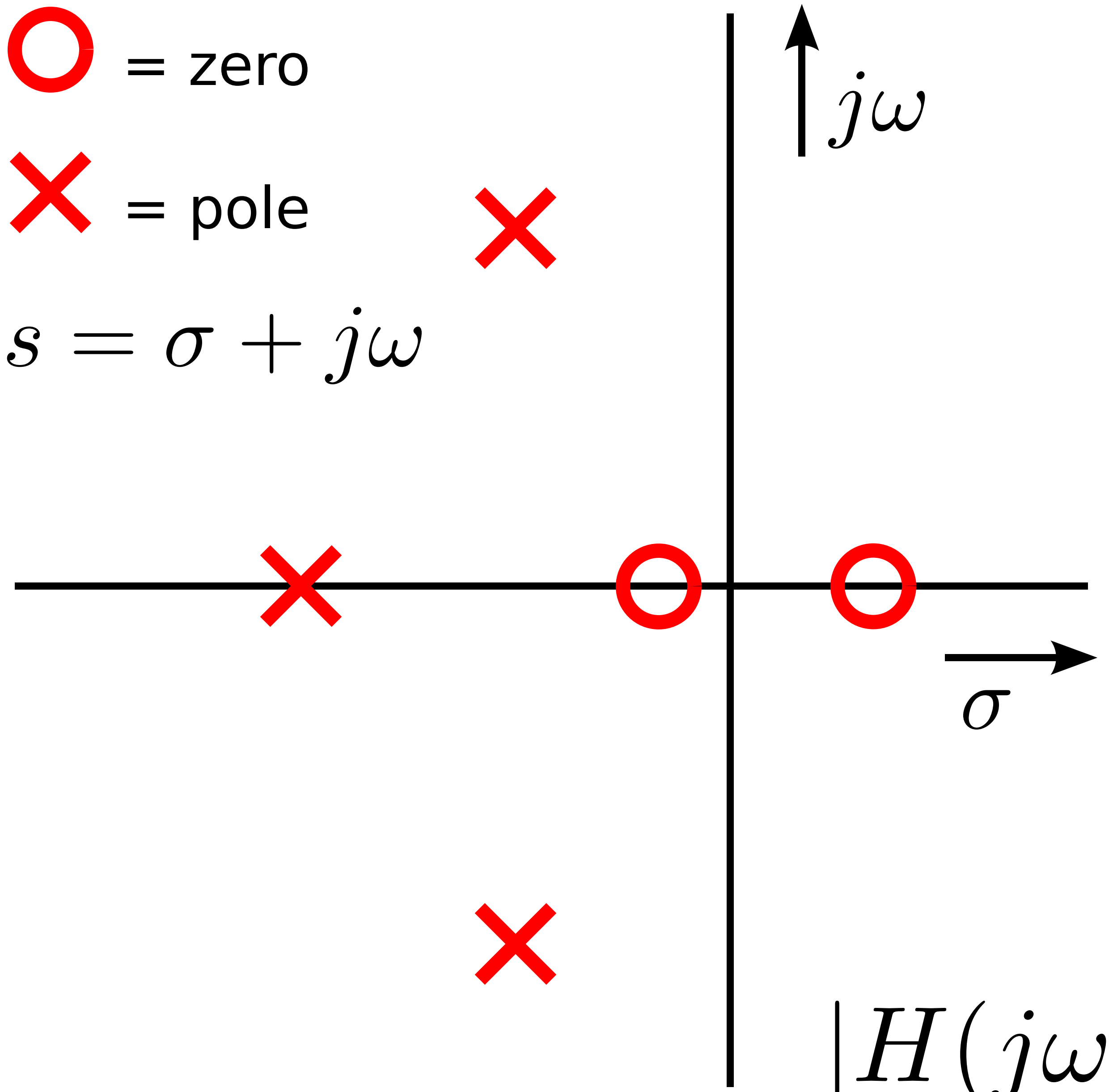
$$\sum_{i=0}^{i=n} a_i s^i = 0$$

$$A = \begin{pmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & \dots & 0 & -a_2 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & 1 & -a_{n-1} \end{pmatrix}$$

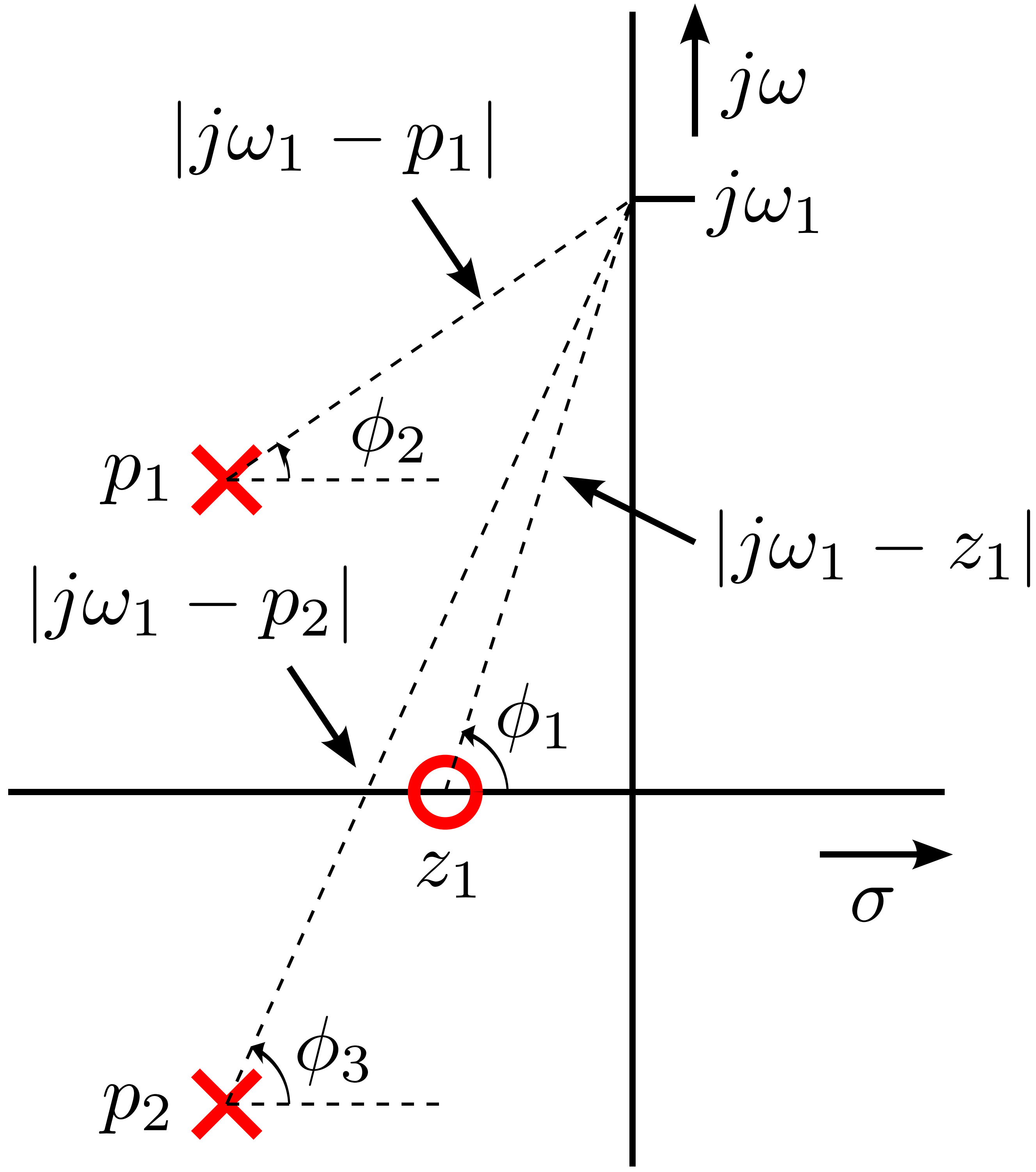
$$\det\left(s\mathrm{I} - \mathrm{A}\right) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$$

Eigenvalues of the companion matrix are the poles of the system.

Pole-zero plots



$$|H(j\omega)| = \frac{b_m}{a_n} \frac{\prod_{k=0}^{k=m} |j\omega - z_k|}{\prod_{i=0}^{i=n} |j\omega - p_i|}$$



$$\arg\{H(j\omega)\} = \arg b_m - \arg a_n + \sum_{k=0}^{k=m} \arg(j\omega - z_k) - \sum_{i=0}^{i=n} \arg(j\omega - p_i)$$

Impulse and step response

Unit impulse response

$$h(t) = \mathcal{L}^{-1}\left\{H(s)\right\}$$

Unit step response

$$a(t) = \int h(t)dt = \mathcal{L}^{-1}\left\{\tfrac{1}{s}H(s)\right\}$$

$$\mathcal{L}^{-1}\left\{H(s)\right\} = \sum_{i=1}^{n}\sum_{\ell=1}^{m}\mathrm{Res}_{i,\ell}\left[e^{st}F\left(s\right)\right]$$

n =number of non coinciding poles

m =number of poles at $s = p_i$

$$\mathrm{Res}_{i,\ell} = \frac{1}{(\ell-1)!}\lim_{s\to p_i}\frac{d^{\ell-1}}{ds^{\ell-1}}\left[\left(s-p_i\right)^{\ell}F\left(s\right)\right]e^{p_i t}$$

stable: $\mathrm{Re}(p_i) < 0\,\forall\, i$

Characterization of LTD systems